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DIFFERENTIAL EQUATIONS ON CLOSED SUBSETS OF A BANACH SPACE(1)

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ABSTRACT. In this paper the problem of existence of solutions to the initial value problem $u'(t) = A(t, u(t))$, $u(a) = z$, is considered where $A : [a, b) \times D \to E$ is continuous, D is a closed subset of a Banach space E, and $z \in D$. With a dissipative type condition on A, we establish sufficient conditions for this initial value problem to have a solution. Using these results, we are able to characterize all continuous functions which are generators of nonlinear semigroups on D.

1. **Introduction.** Suppose that E is a Banach space over the real or complex field and that $|\cdot|$ denotes the norm on E. Let $[a, b)$ be a number interval, D a closed (or locally closed) subset of E, and A a continuous function from $[a, b) \times D$ into E. In this paper we study the existence and uniqueness of solutions to the initial value problem

$$\text{(IVP)} \qquad u' = A(t, u), \qquad u(a) = z,$$

where z is in D. By a solution to (IVP) on an interval $[a, c) \subset [a, b)$, we mean a continuously differentiable function u from $[a, c)$ into D such that $u(a) = z$ and $u'(t) = A(t, u(t))$ for all t in $[a, c)$. Throughout this paper we assume that E has the norm topology.

The main technique employed here is the construction of approximate solutions to (IVP) by using conditions on A similar to those of H. Brézis in [2]. We then place dissipative type conditions on A which ensure the convergence of our approximate solutions and the uniqueness of solutions to (IVP). Using these results we are able to give a characterization of continuous generators of semigroups of nonlinear transformations on D. For other recent papers which deal with similar problems, see F. Browder [4], M. Crandall [5], [6], P. Hartman [8], J. Herod [9], R. Martin [13], [14], R. Redheffer [15], and G. Webb [16], [17].

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Hartman [8], and R. Redheffer [15]. These results indicated several improvements to the original version of this paper.

2. **Approximate solutions.** In this section we place conditions on the function A from $[a,b) \times D$ into E which assure the existence of "approximate" solutions to (IVP). If x is in E we let $d(x;D) = \inf\{|x-y| : y \in D\}$. Also, unless otherwise indicated, we assume that D is locally closed (i.e., for each $x \in D$ there is an $R > 0$ such that $D \cap \{y \in E : |x-y| \leq R\}$ is closed in E). Throughout this paper we frequently assume that the following conditions hold:

(C1) A is continuous from $[a,b) \times D$ into E; and

(C2) $\lim_{h \to 0+} d(x + hA(t,x); D)/h = 0$ for each $(t,x) \in [a,b) \times D$.

Proposition 1. *Suppose that* (C1) *and* (C2) *are fulfilled, z is in D, and R, M and T are positive numbers such that if $F = \{y \in E : |y - z| \leq R\}$, then $D \cap F$ is closed,*

$$\sup\{|A(t,x)| : (t,x) \in [a,b) \times (D \cap F)\} \leq M - 1,$$

and $T < \min\{b - a, R/M\}$. Also, let $\{\varepsilon_n\}_1^\infty$ be a sequence of numbers in $(0,1)$ *such that* $\lim_{n\to\infty} \varepsilon_n = 0$. *Then, for each positive integer n,* (IVP) *has an ε_n-approximate solution u_n from $[a, a+T]$ into F in the following sense: there is a nondecreasing sequence $\{t_i^n\}_{i=0}^\infty$ in $[a, a+T]$ such that*

(i) $t_0^n = a$, $t_i^n < t_{i+1}^n$ *if* $t_i^n < a + T$, $t_{i+1}^n - t_i^n \leq \varepsilon_n$, *and* $\lim_{i\to\infty} t_i^n = a + T$;

(ii) $u_n(a) = z$ *and* $|u_n(t) - u_n(s)| \leq M|t - s|$ *for all* $t, s \in [a, a+T]$;

(iii) $u_n(t_i^n) \in D \cap F$ *for each i and u_n is linear on each of the intervals* $[t_i^n, t_{i+1}^n]$;

(iv) *if* $t_i^n < a + T$ *and* $t \in (t_i^n, t_{i+1}^n)$, *then* $|u_n'(t) - A(t_i^n, u_n(t_i^n))| \leq \varepsilon_n$; *and*

(v) *if* $(t,y) \in [t_i^n, t_{i+1}^n] \times D$ *with* $|y - u_n(t_i^n)| \leq (t_{i+1}^n - t_i^n)M$, *then* $|A(t,y) - A(t_i^n, u_n(t_i^n))| \leq \varepsilon_n$.

Proof. Let $t_0^n = a$ and $u_n(t_0^n) = z$. Inductively define t_{i+1}^n and u_n on $[t_i^n, t_{i+1}^n]$ in the following manner: If $t_i^n = a + T$ let $t_{i+1}^n = a + T$ and if $t_i^n < a + T$ choose $\delta_i^n \in [0, \varepsilon_n]$ such that

(1) $t_i^n + \delta_i^n \leq a + T$;

(2) if $(t,y) \in [t_i^n, t_i^n + \delta_i^n] \times D$ and $|y - u_n(t_i^n)| \leq \delta_i^n M$, then $|A(t,y) - A(t_i^n, u_n(t_i^n))| \leq \varepsilon_n$;

(3) $d(u_n(t_i^n) + \delta_i^n A(t_i^n, u_n(t_i^n)); D) \leq \delta_i^n \varepsilon_n/2$; and

(4) δ_i^n is the largest number in $[0, \varepsilon_n]$ such that (1)–(3) hold.

Note that (Cl) and (C2) ensure that $\delta_i^n > 0$. Define $t_{i+1}^n = t_i^n + \delta_i^n$ and, by (3), let $u_n(t_{i+1}^n)$ be a member of D such that

$$|u_n(t_i^n) + (t_{i+1}^n - t_i^n)A(t_i^n, u_n(t_i^n)) - u_n(t_{i+1}^n)| \leq (t_{i+1}^n - t_i^n)\varepsilon_n, \tag{2.1}$$

and define

$$u_n(t) = [(t - t_i^n)u_n(t_{i+1}^n) + (t_{i+1}^n - t)u_n(t_i^n)]/[t_{i+1}^n - t_i^n] \tag{2.2}$$

for each $t \in (t_i^n, t_{i+1}^n)$. Using (2.1), if t, $s \in [t_i^n, t_{i+1}^n]$, then

$$|u_n(t) - u_n(s)| = (|t - s|)|u_n(t_{i+1}^n) - u_n(t_i^n)|/[t_{i+1}^n - t_i^n]$$
$$\leq |t - s|\{|A(t_i^n, u_n(t_i^n))| + \varepsilon_n\} \leq |t - s|M;$$

so u_n satisfies (ii) on $[a, t_{i+1}^n]$. Also $|u_n(t_{i+1}^n) - z| \leq (t_{i+1}^n - a)M \leq R$, which implies that $u_n(t_{i+1}^n) \in D \cap F$ and that (iii) is fulfilled. If $t \in (t_i^n, t_{i+1}^n)$ then $u_n'(t)$ exists, and by (2.1) and (2.2),

$$|A(t_i^n, u_n(t_i^n)) - u_n'(t)| = |A(t_i^n, u_n(t_i^n)) - [u_n(t_{i+1}^n) - u_n(t_i^n)]/[t_{i+1}^n - t_i^n]|$$
$$\leq \varepsilon_n;$$

so (iv) is also fulfilled. The fact that (v) holds is evident from (2). Assume, for contradiction, that $t_i^n < a + T$ for all $i = 0, 1, 2, \ldots$, and $\lim_{i\to\infty} t_i^n = r < a + T$. Since $|u_n(t_i^n) - u_n(t_j^n)| \leq |t_i^n - t_j^n|M$, $w = \lim_{i\to\infty} u_n(t_i^n)$ exists, and $w \in D \cap F$ since $D \cap F$ is closed. Using (Cl), let $\gamma > 0$ be such that $|A(t,y) - A(r,w)| \leq \varepsilon_n/3$ whenever $|t - r| \leq 2\gamma$ and $|y - w| \leq 2\gamma M$. Using (C2), let $\eta > 0$ be such that $\eta < \min\{\varepsilon_n, \gamma, a + T - r\}$ and

$$d(w + \eta A(r,w); D) \leq \eta\varepsilon_n/3. \tag{2.3}$$

Now let N be sufficiently large so that $r - t_i^n < \eta$ and $|w - u_n(t_i^n)| \leq \eta M$ for all $i \geq N$. Since $\eta \leq \gamma$, if $i \geq N$, $|t - t_i| \leq \eta$, and $|y - u_n(t_i^n)| \leq \eta M$, it follows that $|t - r| \leq 2\eta$, $|y - w| \leq 2\eta M$, and hence

$$|A(t,y) - A(t_i^n, u_n(t_i^n))| \leq |A(t,y) - A(r,w)| + |A(r,w) - A(t_i^n, u_n(t_i^n))|$$
$$\leq 2\varepsilon_n/3.$$

Thus (1) and (2) are valid with δ_i^n replaced by η for each $i \geq N$; so by (4) and the fact that $\eta > \delta_i^n$, it must be true that

$$d(u_n(t_i^n) + \eta A(t_i^n, u_n(t_i^n)); D) > \eta\varepsilon_n/2$$

for all $i \geq N$. However, the function $x \to d(x, D)$ is continuous and it follows that

$$d(w + \eta A(r,w); D) = \lim_{i\to\infty} d(u_n(t_i^n) + \eta A(t_i^n, u_n(t_i^n)); D) \geq \eta\varepsilon_n/2.$$

This is a contradiction to (2.3) and we conclude that $\lim_{i\to\infty} t_i^n = a + T$. Defining $u_n(a + T) = \lim_{i\to\infty} u_n(t_i^n)$ completes the proof of Proposition 1.

We now show that if the approximate solutions constructed in Proposition 1 converge, then they converge to a solution of (IVP).

Proposition 2. *In addition to the suppositions of Proposition* 1, *suppose that* $u(t) = \lim_{n\to\infty} u_n(t)$ *exists for each* t *in* $[a, a + T]$. *Then* u *is a solution to* (IVP) *on* $[a, a + T)$.

Proof. Since the sequence $\{u_n\}_1^\infty$ is equicontinuous on $[a, a+T]$ by (ii) of Proposition 1, it follows that $\{u_n\}_1^\infty$ converges uniformly to u on $[a, a+T]$ and that u is continuous. Also, if $t \in [a, a+T)$ and, for each n, i_n is such that $t \in [t_{i_n}^n, t_{i_n+1}^n)$, then

$$\lim_{n\to\infty} |u(t) - u_n(t_{i_n}^n)| \leq \lim_{n\to\infty} \{|u(t) - u_n(t)| + |t - t_{i_n}^n| M\} = 0. \tag{2.4}$$

Since $u_n(t_{i_n}^n) \in D \cap F$ and $D \cap F$ is closed, we have that $u(t) \in D \cap F$. Hence u maps $[a, a+T]$ into D. Since the set $K = \{(t, u(t)) : t \in [a, a+T]\}$ is a compact subset of $[a, a+T] \times D$ and $|u_n'(t) - A(t_i^n, u_n(t_i^n))| \leq \varepsilon_n$ whenever $t \in (t_i^n, t_{i+1}^n)$ by (iv) of Proposition 1, it follows easily from (2.4) and the continuity of A that $\lim_{n\to\infty} u_n'(t) = A(t, u(t))$, uniformly on $[a, a+T] - S$ where S is a countable subset of $[a, a+T]$. Consequently, for each t in $[a, a+T]$,

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \left\{ z + \int_a^t u_n'(s)\, ds \right\} = z + \int_a^t A(s, u(s))\, ds.$$

Thus u is a solution to (IVP) and Proposition 2 is proved.

3. **Existence of solutions.** In this section we use the results of §2 and a dissipative type condition on A to establish criteria which guarantee the existence and uniqueness of solutions to (IVP). Let E^* denote the dual space of E and let J be the duality mapping from E into the class of subsets of E^* (i.e., for each x in E, $Jx = \{x^* \in E^* : x^*(x) = |x|^2 = |x^*|^2\}$). For each x and y in E define

$$\begin{aligned} \langle x, y \rangle_- &= \inf\{\operatorname{Re} x^*(y) : x^* \in Jx\} \quad \text{and} \\ \langle x, y \rangle_+ &= \sup\{\operatorname{Re} x^*(y) : x^* \in Jx\}. \end{aligned} \tag{3.1}$$

Note that if x, y and z are in E then

$$\langle x, y+z \rangle_- \leq \langle x, y \rangle_- + |x||z| \quad \text{and} \quad \langle x, y+z \rangle_+ \leq \langle x, y \rangle_+ + |x||z|. \tag{3.2}$$

Furthermore, if u is a function from $[a, b)$ into E which is differentiable at $t_0 \in (a, b)$, and $p(t) = |u(t)|^2$ for all t in $[a, b)$, then p is both left and right differentiable at t_0 with

$$p_-'(t_0) = 2\langle u(t_0), u'(t_0) \rangle_- \quad \text{and} \quad p_+'(t_0) = 2\langle u(t_0), u'(t_0) \rangle_+ \tag{3.3}$$

(see T. Kato [10, Lemma 1.3]). We use the properties (3.2) and (3.3) frequently and without comment in the proofs of our results.

Theorem 1. *In addition to the suppositions of Proposition* 1, *suppose that* $D \cap F$ *is convex and there is a number L such that*

$$\langle x - y, A(t, x) - A(t, y) \rangle_- \leq L|x - y|^2$$

for all (t, x) *and* (t, y) *in* $[a, a+T] \times (D \cap F)$. *Then* (IVP) *has a unique solution on* $[a, a+T]$.

Proof. Using the fact that $D \cap F$ is convex and part (iii) of Proposition 1, we have that $u_n(t) \in D \cap F$ for all $t \in [a, a+T]$. Let n and m be positive integers and define $p(t) = |u_n(t) - u_m(t)|^2$ for each t in $[a, a+T]$. If $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$, then

$$p'_-(t) = 2\langle u_n(t) - u_m(t), u'_n(t) - u'_m(t)\rangle_- .$$

However, $|u_n(t) - u_n(t_i^n)| \leq (t - t_i^n)M$, so by parts (iv) and (v) of Proposition 1,

$$\begin{aligned}|u'_n(t) - A(t, u_n(t))| &\leq |u'_n(t) - A(t_i^n, u_n(t_i^n))| + |A(t_i^n, u_n(t_i^n)) - A(t, u_n(t))| \\ &\leq 2\varepsilon_n .\end{aligned}$$

Similarly, $|u'_m(t) - A(t, u_m(t))| \leq 2\varepsilon_m$. Using the above inequalities we have that

$$\begin{aligned}p'_-(t) &\leq 2\langle u_n(t) - u_m(t), A(t, u_n(t)) - A(t, u_m(t))\rangle_- \\ &\quad + 4|u_n(t) - u_m(t)|(\varepsilon_n + \varepsilon_m) \\ &\leq 2Lp(t) + 8(R + |z|)(\varepsilon_n + \varepsilon_m).\end{aligned}$$

Since this inequality holds for all but a countable number of t in $[a, a+T]$ and $p(a) = 0$, it follows that

$$|u_n(t) - u_m(t)|^2 \leq 8(R + |z|)(\varepsilon_n + \varepsilon_m) \int_a^t e^{2L(t-s)}\, ds$$

for all $t \in [a, a+T]$. Thus the sequence $\{u_n\}_1^\infty$ is uniformly Cauchy on $[a, a+T]$ and the existence of solutions to (IVP) follows from Proposition 2. To establish uniqueness, note that if u and v are solutions and $p(t) = |u(t) - v(t)|^2$, then

$$p'_-(t) = 2\langle u(t) - v(t), A(t, u(t)) - A(t, v(t))\rangle_- \leq 2Lp(t);$$

so $p(t) = 0$ and Theorem 1 is established.

A crucial point in the proof of Theorem 1 was the fact that $D \cap F$ is convex, and hence the approximate solutions have range in $D \cap F$. In general, this is not the case and we must use a slightly different approach. Our first result in this direction uses a condition due to M. Crandall [5]. For a closely related condition, see R. Martin [13].

Theorem 2. *In addition to the suppositions of Proposition* 1, *suppose that there is a number L and a nondecreasing function* $g : [0, \infty) \to [0, \infty)$ *such that* $\lim_{r \to 0+} g(r) = 0$ *and*

$$\text{(3.4)} \qquad \begin{aligned}&\langle x - y, A(t, x_1) - A(s, y_1)\rangle_- \\ &\quad \leq g(|t - s| + |x - x_1| + |y - y_1|) + L|x - y|^2\end{aligned}$$

for all $(t, x_1), (s, y_1) \in [a, a+T] \times (D \cap F)$ *and all* $x, y \in F$ *Then* (IVP) *has a unique solution on* $[a, a+T]$.

Remark 1. If it is assumed that $\langle x - y, A(t,x) - A(t,y)\rangle_- \leq L|x - y|^2$ for all (t,x), $(t,y) \in [a, a + T] \times (D \cap F)$ and either (a) E^* is Lipschitz convex, or (b) A is Lipschitz continuous on $[a, a + T] \times (D \cap F)$, then (3.4) is satisfied.

Proof of Theorem 2. Let n and m be positive integers and define $p(t) = |u_n(t) - u_m(t)|^2$ for $t \in [a, a + T]$. By part (iv) of Proposition 1, if $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$ then

$$\begin{aligned} p'_-(t) &= 2\langle u_n(t) - u_m(t), u'_n(t) - u'_m(t)\rangle_- \\ &\leq 2\langle u_n(t) - u_m(t), A(t_i^n, u_n(t_i^n)) - A(t_j^m, u_m(t_j^m))\rangle_- \\ &\quad + 2|u_n(t) - u_m(t)|(\varepsilon_n + \varepsilon_m). \end{aligned}$$

Using the facts that $|u_n(t) - u_m(t)| \leq 2(R + |z|)$, $|t_i^n - t_j^m| \leq \varepsilon_n + \varepsilon_m$, $|u_n(t) - u_n(t_i^n)| \leq \varepsilon_n M$, and $|u_m(t) - u_m(t_j^m)| \leq \varepsilon_m M$, it follows from (3.4) that

$$p'_-(t) \leq 2Lp(t) + 2g((\varepsilon_n + \varepsilon_m)(1 + M)) + 4(R + |z|)(\varepsilon_n + \varepsilon_m).$$

Solving this differential inequality and noting that $p(a) = 0$ and

$$\lim_{n,m\to\infty}\{2g((\varepsilon_n + \varepsilon_m)(1 + M)) + 4(R + |z|)(\varepsilon_n + \varepsilon_m)\} = 0$$

establishes the fact that $\{u_n\}_1^\infty$ is uniformly Cauchy on $[a, a + T]$; so (IVP) has a solution by Proposition 2. The uniqueness assertion follows as in the proof of Theorem 1.

For the characterization of continuous generators of nonlinear semigroups, we need a result which is closely related to Theorems 1 and 2. However, the proof is tedious and uses the fact that the function $(x,y) \to \langle x,y\rangle_+$ from $E \times E$ into $(-\infty, \infty)$ is upper semicontinuous (i.e., if $\lim_{n\to\infty}(x_n, y_n) = (x,y)$ then $\limsup_{n\to\infty}\langle x_n, y_n\rangle_+ \leq \langle x,y\rangle_+$—see M. Crandall and T. Liggett [7, Lemma 2.16]).

Theorem 3. *In addition to the suppositions of Proposition* 1, *suppose that there is a number L such that*

$$\langle x - y, A(t,x) - A(t,y)\rangle_+ \leq L|x - y|^2 \tag{3.5}$$

for all (t,x) *and* (t,y) *in* $[a, a + T] \times (D \cap F)$. *Then* (IVP) *has a unique solution on* $[a, a + T)$.

For the proof of Theorem 3 we use two preliminary lemmas, each of which is with the suppositions of Theorem 3. Let m and n be positive integers and let $S = \bigcup_{i=1}^\infty \{t_i^n, t_i^m\}$. Inductively, define the sequence $\{r_k\}_{k=0}^\infty$ in $[a, a + T]$ by $r_0 = a$ and $r_{k+1} = \min\{s \in S : s > r_k\}$ for each $k = 0, 1, 2, \ldots$. Note that $\{r_k\}_1^\infty$ is the "minimal refinement" of the sequences $\{t_i^n\}_{i=1}^\infty$ and $\{t_j^m\}_{j=1}^\infty$.

Lemma 1. *With the notations of the above paragraph, there exist functions* v_n *and* v_m *from* $[a, a + T]$ *into F with the following properties*:

(i) *For* $\beta \in \{m,n\}$ *and* k *a nonnegative integer,* $|v_\beta(t) - z| \leq (t-a)M$ *and* $|v_\beta(t) - v_\beta(s)| \leq |t-s|M$ *for all* $t, s \in [r_k, r_{k+1})$;

(ii) $v_n(r_{k+1})$ *and* $v_m(r_{k+1})$ *are in* $D \cap F$;

(iii) *for all but a countable number of* $t \in [r_k, r_{k+1})$, $v_n'(t)$ *and* $v_m'(t)$ *exist with*

$$\langle v_n(t) - v_m(t), v_n'(t) - v_m'(t)\rangle_+ \leq L|v_n(t) - v_m(t)|^2 + (\varepsilon_n + \varepsilon_m)[1 + |v_n(t) - v_m(t)|];$$

(iv) *if* k *is a nonnegative integer and* i *and* j *are integers such that* $t_i^n, t_j^m \leq r_k \leq r_{k+1} \leq t_{i+1}^n, t_{j+1}^m$, *then*

(a) $v_n(r_{k+1}) = u_n(r_{k+1})$ *if* $r_{k+1} = t_{i+1}^n$ *and* $v_n(r_{k+1}) = v_n(r_{k+1}-)$ *if* $r_{k+1} < t_{i+1}^n$,

(b) $v_m(r_{k+1}) = u_m(r_{k+1})$ *if* $r_{k+1} = t_{j+1}^m$ *and* $v_m(r_{k+1}) = v_m(r_{k+1}-)$ *if* $r_{k+1} < t_{j+1}^m$, *and*

(c) $|v_n(t) - u_n(t)| \leq 3(t - t_i^n)\varepsilon_n$ *and* $|v_m(t) - u_m(t)| \leq 3(t - t_j^m)\varepsilon_m$ *for all* $t \in [r_k, r_{k+1}]$; *and*

(v) *if* i *is a nonnegative integer and* $\beta \in \{m,n\}$ *then*

$$|v_\beta(t_{i+1}^\beta) - v_\beta(t_{i+1}^\beta -)| \leq 3(t_{i+1}^\beta - t_i^\beta)\varepsilon_\beta.$$

Proof. The proof of this lemma uses techniques similar to the proof of Proposition 1. We assume that k is a nonnegative integer, v_n and v_m are defined on $[a, r_k]$ and satisfy the properties listed on the interval $[a, r_k]$. Hence we need to show that v_n and v_m can be appropriately extended to $[a, r_{k+1}]$. For the construction of v_n and v_m on $[r_k, r_{k+1}]$, let $s_0 = r_k$ and inductively define the sequence $\{s_l\}_0^\infty$ and v_n and v_m on $[r_k, s_{l+1}]$ as follows: if $s_l = r_{k+1}$ then $s_{l+1} = r_{k+1}$ and if $s_l < r_{k+1}$ let $s_{l+1} = s_l + \gamma_l$ where $\gamma_l \geq 0$ is such that

(1)′ $s_l + \gamma_l \leq r_{k+1}$;

(2)′ $d(v_\beta(s_l) + \gamma_l A(s_l, v_\beta(s_l)); D) \leq \gamma_l \varepsilon_\beta/2$ for $\beta \in \{m,n\}$;

(3)′
$$\langle (v_n(s_l) + x) - (v_m(s_l) + y), A(s_l, v_n(s_l)) - A(s_l, v_m(s_l))\rangle_+ \leq L|(v_n(s_l) + x) - (v_m(s_l) + y)|^2 + \varepsilon_n + \varepsilon_m$$

whenever $|x|, |y| \leq \gamma_l M$; and

(4)′ γ_l is the largest number such that (1)′–(3)′ hold.

It follows from (C2), (3.5) and the upper semicontinuity of $\langle \cdot, \cdot \rangle_+$ that $\gamma_l > 0$. Using (2)′ for each $\beta \in \{m,n\}$ let $v_\beta(s_{l+1})$ be a member of D such that

(3.6) $$|v_\beta(s_l) + (s_{l+1} - s_l)A(s_l, v_\beta(s_l)) - v_\beta(s_{l+1})| \leq (s_{l+1} - s_l)\varepsilon_\beta,$$

and for each $t \in (s_l, s_{l+1})$ define

$$v_\beta(t) = [(t - s_l)v_\beta(s_{l+1}) + (s_{l+1} - t)v_\beta(s_l)]/[s_{l+1} - s_l].$$

It is straightforward to see that $v_\beta(s_{l+1}) \in D \cap F$, $|v_\beta(t) - v_\beta(s)| \leq |t-s|M$, and $|v_\beta(t) - z| \leq (t-a)M$ for all $t, s \in [r_k, s_{l+1}]$. In particular, (i) holds for $t \in [r_k, s_{l+1})$. Furthermore, by (3.6) and the definition of v_β, (3.7) $$|v'_\beta(t) - A(s_l, v_\beta(s_l))| \leq \varepsilon_\beta$$

for $t \in (s_l, s_{l+1})$. Since $|v_\beta(t) - v_\beta(s_l)| \leq \gamma_l M$ for $t \in (s_l, s_{l+1})$, it follows from (3)′ and (3.7) that

$$\begin{aligned}\langle v_n(t) - v_m(t), v'_n(t) - v'_m(t)\rangle_+ &\\ &\leq \langle v_n(t) - v_m(t), A(s_l, v_n(s_l)) - A(s_l, v_m(s_l))\rangle_+ \\ &\quad + |v_n(t) - v_m(t)|(\varepsilon_n + \varepsilon_m) \\ &\leq L|v_n(t) - v_m(t)|^2 + (\varepsilon_n + \varepsilon_m)(1 + |v_n(t) - v_m(t)|)\end{aligned}$$

for all $t \in (s_l, s_{l+1})$. Thus (iii) holds for $t \in [r_k, s_{l+1})$. Now assume, for contradiction, that $\lim_{l\to\infty} s_l = \tau < r_{k+1}$ and let $w_\beta = \lim_{l\to\infty} v_\beta(s_l)$ for $\beta \in \{m, n\}$. Then $w_\beta \in D \cap F$ so

$$\langle w_n - w_m, A(\tau, w_n) - A(\tau, w_m)\rangle_+ \leq L|w_n - w_m|^2.$$

Using the upper semicontinuity of $\langle\cdot,\cdot\rangle_+$ and (C2), one can find an $\eta > 0$ such that $\eta < r_{k+1} - \tau$;

(3.8) $$\begin{aligned}&\langle (w_n + x_n) - (w_m + x_m), [A(\tau, w_n) + y_n] - [A(\tau, w_m) + y_m]\rangle_+ \\ &\qquad \leq L|(w_n + x_n) - (w_m + x_m)|^2 + (\varepsilon_n + \varepsilon_m)/2\end{aligned}$$

whenever $|x_n|, |x_m| \leq 2\eta M$ and $|y_n|, |y_m| \leq \eta$; and

(3.9) $$d(w_\beta + \eta A(\tau, w_\beta); D) \leq \eta\varepsilon_\beta/3$$

for $\beta \in \{m, n\}$. Let N be sufficiently large so that if $l \geq N$ then $s_{l+1} - s_l < \eta$, $|v_\beta(s_l) - w_\beta| \leq \eta M$, and $|A(s_l, v_\beta(s_l)) - A(\tau, w_\beta)| \leq \eta$. If $|x|, |y| \leq \eta M$, we have from (3.8) and the choice of η that (3)′ holds with γ_l replaced by η for each $l \geq N$. Since γ_l was chosen maximally, it follows that for each $l \geq N$, either

$$d(v_n(s_l) + \eta A(s_l, v_n(s_l)); D) > \eta\varepsilon_n/2 \quad \text{or}$$

$$d(v_m(s_l) + \eta A(s_l, v_m(s_l)); D) > \eta\varepsilon_m/2.$$

Using the fact that $x \to d(x; D)$ is continuous it follows that either

$$d(w_n + \eta A(\tau, w_n); D) \geq \eta\varepsilon_n/2 \quad \text{or} \quad d(w_m + \eta A(\tau, w_m); D) \geq \eta\varepsilon_m/2.$$

This is a contradiction to (3.9) and we conclude that $\lim_{l\to\infty} s_l = r_{k+1}$. Now let i and j be integers such that $t_i^n, t_j^m \leq r_k < r_{k+1} \leq t_{i+1}^n, t_{j+1}^m$. Since $v_\beta(r_{k+1}-) = \lim_{l\to\infty} v_\beta(s_l)$ and $D \cap F$ is closed, we have $v_\beta(r_{k+1}-) \in D \cap F$ for $\beta \in \{m, n\}$. If $r_{k+1} < t_{i+1}^n$ let $v_n(r_{k+1}) = v_n(r_{k+1}-)$ and if $r_{k+1} = t_{i+1}^n$ let $v_n(r_{k+1}) = u_n(r_{k+1})$. Defining $v_m(r_{k+1})$ analogously, we have that (ii) is satisfied, as well as parts (a) and (b) of (iv). Since $v_n(t_i^n) = u_n(t_i^n)$ and v_n is continuous on $[t_i^n, r_{k+1})$, it follows from (i) that $|v_n(t) - u_n(t_i^n)| \leq (t - t_i^n)M$; so by part (v) of Proposition 1,

$$|A(s_l, v_n(s_l)) - A(t_i^n, u_n(t_i^n))| \leq \varepsilon_n$$

for all $l \geq 0$. Thus, by (3.7) and (iv) of Proposition 1, if $t \in (s_l, s_{l+1})$ then

$$|v_n'(t) - u_n'(t)| \leq |v_n'(t) - A(s_l, v_n(s_l))| + |A(s_l, v_n(s_l)) - A(t_i^n, u_n(t_i^n))| + |A(t_i^n, u_n(t_i^n)) - u_n'(t)| \leq 3\varepsilon_n.$$

Consequently, if $t \in [r_k, r_{k+1}]$,

$$|v_n(t) - u_n(t)| \leq |v_n(r_k) - u_n(r_k)| + \int_{r_k}^{t} |v_n'(s) - u_n'(s)|\, ds \leq 3(r_k - t_i^n)\varepsilon_n + 3(t - r_k)\varepsilon_n = 3(t - t_i^n)\varepsilon_n.$$

The fact that $|v_m(t) - u_m(t)| \leq 3(t - t_j^m)\varepsilon_m$ is proved analogously and part (c) of (iv) is established. Let i be a nonnegative integer and let $\beta \in \{n, m\}$. Using part (iv), we have that

$$\begin{aligned} |v_\beta(t_{i+1}^\beta) - v_\beta(t_{i+1}^\beta -)| &= |u_\beta(t_{i+1}^\beta) - v_\beta(t_{i+1}^\beta -)| \\ &= \lim_{h\to 0+} |u_\beta(t_{i+1}^\beta - h) - v_\beta(t_{i+1}^\beta - h)| \\ &\leq \lim_{h\to 0+} 3(t_{i+1}^\beta - h - t_i^\beta)\varepsilon_\beta = 3(t_{i+1}^\beta - t_i^\beta)\varepsilon_\beta. \end{aligned}$$

This establishes (v) and the proof of Lemma 1 is complete.

Lemma 2. *Suppose that Λ and ε are positive numbers and p is a function from $[a, a+T]$ into $[0, \infty)$ such that*

(i) *p is continuous on $[r_k, r_{k+1})$ for each $k \geq 0$;*

(ii) *$p(a) = 0$ and $p(r_{k+1} -)$ exists for each $k \geq 0$; and*

(iii) *$p'(t)$ exists and $p'(t) \leq \Lambda p(t) + \varepsilon$ for all but a countable number of t in $[a, a+T]$.*

If $k \geq 0$ and t is in $[r_k, r_{k+1})$, then

$$p(t) \leq \left[\varepsilon\Lambda^{-1} + \sum_{i=1}^{k} |p(r_i) - p(r_i -)|\right] e^{\Lambda(t-a)} - \varepsilon\Lambda^{-1}.$$

Proof. Since p is continuous on $[r_0, r_1)$, the inequality is immediate for $k = 0$ (where $\sum_{i=1}^{0} |p(r_i) - p(r_i -)| = 0$). Assume the inequality is valid for $t \in [r_{k-1}, r_k)$. If $t \in [r_k, r_{k+1})$, then p is continuous on $[r_k, r_{k+1})$ and it follows that

$$p(t) \leq p(r_k)\exp(\Lambda(t - r_k)) + [\exp(\Lambda(t - r_k)) - 1]\varepsilon\Lambda^{-1}. \tag{3.10}$$

Also,

$$\begin{aligned} p(r_k) &\leq p(r_k -) + |p(r_k) - p(r_k -)| \\ &\leq \left[\varepsilon\Lambda^{-1} + \sum_{i=1}^{k-1} |p(r_i) - p(r_i -)|\right]\exp(\Lambda(r_k - a)) - \varepsilon\Lambda^{-1} \\ &\quad + |p(r_k) - p(r_k -)| \\ &\leq \left[\varepsilon\Lambda^{-1} + \sum_{i=1}^{k} |p(r_i) - p(r_i -)|\right]\exp(\Lambda(r_k - a)) - \varepsilon\Lambda^{-1}. \end{aligned}$$

Substituting this estimate into (3.10) establishes the assertion of Lemma 2 by induction.

Proof of Theorem 3. Let n and m be positive integers, let v_n and v_m be as in Lemma 1, and let $p(t) = |v_n(t) - v_m(t)|^2$ for $t \in [a, a+T]$. By part (iii) of Lemma 1,

$$p'_+(t) \leq 2Lp(t) + 2(\varepsilon_m + \varepsilon_n)(1 + |v_n(t) - v_m(t)|)$$

for all but a countable number of $t \in [a, a+T]$. Furthermore, $|v_n(t) - v_m(t)| \leq 2(R + |z|)$ and for each $k \geq 1$,

$$\begin{aligned} |p(r_k) - p(r_k -)| &= [|v_n(r_k) - v_m(r_k)| + |v_n(r_k -) - v_m(r_k -)|] \\ &\quad \cdot [|v_n(r_k) - v_m(r_k)| - |v_n(r_k -) - v_m(r_k -)|] \\ &\leq 4(R + |z|)[|v_n(r_k) - v_n(r_k -)| + |v_m(r_k) - v_m(r_k -)|]. \end{aligned}$$

Using parts (iv) and (v) of Lemma 1, it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} |p(r_k) - p(r_k -)| \\ &\quad \leq 4(R + |z|) \left\{ \sum_{i=0}^{\infty} |v_n(t_{i+1}^n) - v_n(t_{i+1}^n -)| + \sum_{j=0}^{\infty} |v_m(t_{j+1}^m) - v_m(t_{j+1}^m -)| \right\} \\ &\quad \leq 4(R + |z|) \left\{ \sum_{i=0}^{\infty} 3(t_{i+1}^n - t_i^n)\varepsilon_n + \sum_{j=0}^{\infty} 3(t_{j+1}^m - t_j^m)\varepsilon_m \right\} \\ &\quad = 12(R + |z|)T(\varepsilon_n + \varepsilon_m). \end{aligned}$$

Applying Lemma 2, we see that $|v_n(t) - v_m(t)|^2 \leq \delta_{n,m}$ where $\lim_{n,m\to\infty} \delta_{n,m} = 0$. Hence $|u_n(t) - u_m(t)|^2 \leq \alpha_{n,m}$ where $\lim_{n,m\to\infty} \alpha_{n,m} = 0$ by part (iv) (c) of Lemma 1. Therefore $\{u_n\}_1^\infty$ is uniformly Cauchy on $[a, a+T]$ and (IVP) has a solution by Proposition 2. The uniqueness assertion follows as in the proof of Theorem 1 and the proof of Theorem 3 is complete.

Concerning the existence of solutions to (IVP) on the interval $[a, b)$ we have

Theorem 4. *Suppose that D is closed, A satisfies conditions* (C1) *and* (C2), *and there is a continuous real valued function ρ defined on $[a, b)$ such that*

$$\langle x - y, A(t,x) - A(t,y) \rangle_+ \leq \rho(t)|x - y|^2 \tag{3.11}$$

for all (t,x) and (t,y) in $[a,b) \times D$. Suppose further that at least one of the following is fulfilled: (i) *for each $c \in [a,b)$ and $R > 0$, there is a number $M(c,R) > 0$ such that $|A(t,x)| \leq M(c,R)$ for all (t,x) in $[a,c] \times D$ with $|x| \leq R$; or* (ii) *for each $c \in [a,b)$, $R > 0$ and $\varepsilon > 0$, there is a number $\delta(c,\varepsilon,R) > 0$ such that $|A(t,x) - A(s,x)| \leq \varepsilon$ whenever (t,x) and (s,x) are in $[a,c] \times D$ with $|t - s| \leq \delta(c,\varepsilon,R)$*

and $|x| \leq R$. Then for each z in D there is a unique solution u_z to (IVP) *on* $[a, b)$. *Furthermore, if z and w are in D then*

$$|u_z(t) - u_w(t)| \leq |z - w| \exp\left(\int_a^t \rho(s)\, ds \right)$$

for all $t \in [a, b)$.

Remark 3. If, in Theorem 4, it is assumed that D is also convex, then (3.11) can be replaced by the inequality

$$\langle x - y, A(t,x) - A(t,y) \rangle_- \leq \rho(t)|x - y|^2$$

for all (t, x) and (t, y) in $[a, b) \times D$. Also, in the case that $D = E$, it is shown by D. Lovelady and R. Martin [12] that (Cl), (C2) and (3.11) are sufficient to ensure the existence of solutions on $[a, b)$. However, the author does not know if (Cl), (C2) and (3.11) are sufficient in general.

Proof of Theorem 4. Note first that by Theorem 3, for each z in D there is a unique function u_z which is a solution to (IVP) on $[a, b_z)$. Also, we can assume that u_z is noncontinuable. Assume, for contradiction, that $b_z < b$. If $p(t) = |u_z(t) - z|^2$ for $t \in [a, b_z]$ then

$$\begin{aligned} p'_+(t) &= 2\langle u_z(t) - z, A(t, u_z(t)) \rangle_+ \\ &\leq 2\langle u_z(t) - z, A(t, u_z(t)) - A(t, z) \rangle_+ + 2|u_z(t) - z||A(t, z)| \\ &\leq 2\rho(t)p(t) + 2p(t)^{1/2}|A(t, z)|. \end{aligned}$$

Since $q(t) = p(t)^{1/2}$ is right differentiable and $q'_+(t) = 0$ if $p(t) = 0$, we see that $q'_+(t) \leq \rho(t)q(t) + |A(t, z)|$ for each t in $[a, b_z)$. Solving this differential inequality shows that there is a number $R > 0$ such that $|u_z(t)| \leq R$ for all t in $[a, b_z)$. If (i) holds then $|u_z(t) - u_z(s)| \leq M(b_z, R)|t - s|$ for all $s, t \in [a, b_z)$; so $\lim_{t \to b_z} u_z(t)$ exists and is in D, since D is closed. This is a contradiction to the fact that u_z is noncontinuable. If (ii) holds, ε and h are positive numbers with $h \leq \delta(b_z, \varepsilon, R)$, and $p(t) = |u_z(t + h) - u_z(t)|^2$ for $t \in [a, b_z - h)$, then

$$\begin{aligned} p'_+(t) &= 2\langle u_z(t + h) - u_z(t), A(t + h, u_z(t + h)) - A(t, u_z(t)) \rangle_+ \\ &\leq 2\rho(t)p(t) + 2p(t)^{1/2}|A(t + h, u_z(t + h)) - A(t, u_z(t + h))| \\ &\leq 2\rho(t)p(t) + 2p(t)^{1/2}\varepsilon. \end{aligned}$$

Thus if $q(t) = p(t)^{1/2}$, then $q'_+(t) \leq \rho(t)q(t) + \varepsilon$. Solving this differential inequality shows that $\lim_{t \to b_z} u_z(t)$ exists and is in D, again contradicting the fact that u_z is noncontinuable. Consequently, $b_z = b$ for each z in D. If z and w are in D and $p(t) = |u_z(t) - u_w(t)|^2$ for $t \in [a, b)$, then $p'_+(t) \leq 2\rho(t)p(t)$ and we have that

$$|u_z(t) - u_w(t)|^2 \le |z - w|^2 \exp\left(2 \int_a^t \rho(s)\, ds\right)$$

for all $t \in [a, b)$. This completes the proof of Theorem 4.

4. **Semigroups of nonlinear operators.** In this section we apply our results to the theory of nonlinear semigroups. If ω is a real number, D is a closed subset of E, and $U = \{U(t) : t \ge 0\}$ is a family of functions from D into D, then U is said to be a semigroup of type ω on D if each of the following are fulfilled: (a) $U(0)x = x$; (b) $U(t + s) = U(t)U(s)x$; (c) $|U(t)x - U(t)y| \le |x - y|e^{\omega t}$; and (d) $\lim_{h\to 0+} U(h)x = x$, for all $t, s \in [0, \infty)$ and $x, y \in D$. For each $h > 0$ and $x \in D$ let $A^h x = [U(h)x - x]/h$, $D(A) = \{x \in D : \lim_{h\to 0+} A^h x \text{ exists}\}$, and $Ax = \lim_{h\to 0+} A^h x$ for all $x \in D(A)$. A is called the (strong) generator of U.

Theorem 5. *Suppose that D is a closed subset of E and A is a continuous function from D into E. Then these are equivalent*:

(i) *A is the generator of a semigroup U of type ω on D.*

(ii) $\langle x - y, Ax - Ay\rangle_+ \le \omega|x - y|^2$ *and* $\lim_{h\to 0+} d(x + hAx; D)/h = 0$ *for each x and y in D.*

Proof. Suppose that (i) holds, $x, y \in D$, and $p(t) = |U(t)x - U(t)y|^2$ for all $t \ge 0$. Then

$$\begin{aligned}\langle x - y, Ax - Ay\rangle_+ &= 2^{-1}p'_+(0)\\ &= 2^{-1}\lim_{h\to 0+}[|U(h)x - U(h)y|^2 - |x - y|^2]/h\\ &\le 2^{-1}\lim_{h\to 0+}[e^{2\omega h}|x - y|^2 - |x - y|^2]/h\\ &= \omega|x - y|^2,\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0+} d(x + hAx; D)/h &\le \lim_{h\to 0+}|x + hAx - U(h)x|/h\\ &= \lim_{h\to 0+}|Ax - A^h x| = 0.\end{aligned}$$

Thus (i) implies (ii). Conversely, if (ii) holds, then the suppositions of Theorem 4 are fulfilled with $[a, b) = [0, \infty)$ and $A(t, x) = Ax$ for all $(t, x) \in [0, \infty) \times D$ (note the assumption (ii) of Theorem 4 holds in this case); so for all $t \ge 0$ let $U(t)z = u_z(t)$ where u_z is the solution to (IVP) for each $z \in D$. It is easy to see that U is a semigroup of type ω on D and A is the generator of U. Thus (ii) implies (i) and the proof of Theorem 5 is complete.

In the case that D is a closed and convex subset of E, we have the following further characterization of continuous generators:

Theorem 6. *Suppose that D is a closed and convex subset of E and A is a continuous function from D into E. Then these are equivalent:*

(i) *A is the generator of a semigroup U of type ω on D.*

(ii) *$\langle x - y, Ax - Ay\rangle_- \leq \omega|x - y|^2$ and $\lim_{h\to 0+} d(x + hAx; D)/h = 0$ for each x and y in D.*

(iii) *$\langle x - y, Ax - Ay\rangle_- \leq \omega|x - y|^2$ for each x and y in D, and for each $\varepsilon > 0$ such that $\varepsilon\omega < 1$, the image of D under the mapping $I - \varepsilon A : x \to x - \varepsilon Ax$ contains D.*

Proof. The proof of the equivalence of (i) and (ii) is the same as the proof of Theorem 5 (see, in particular, Remark 3). Suppose that (iii) holds and let $h > 0$ be such that $h\omega < 1$. If $x \in D$ then $|(I - hA)^{-1}x - x| \leq h(1 - h\omega)^{-1}|Ax|$ (see, e.g., [7, Lemma 1.2]), and hence $\lim_{h\to 0+} A(I - hA)^{-1}x = Ax$ by the continuity of A. Consequently, noting that $A(I - hA)^{-1}x = [(I - hA)^{-1}x - x]/h$,

$$\begin{aligned} d(x + hAx; D)/h &\leq |x + hAx - (I - hA)^{-1}x|/h \\ &= |Ax - [(I - hA)^{-1}x - x]/h| \\ &= |Ax - A(I - hA)^{-1}x|; \end{aligned}$$

and it follows that $\lim_{h\to 0+} d(x + hAx; D)/h = 0$. Thus (iii) implies (ii). Now suppose that (ii) holds and let w be in D and ε a positive number such that $\varepsilon\omega < 1$. For each x in D define $Bx = \varepsilon Ax + w - x$. Then B is continuous on D and if x and y are in D,

$$\begin{aligned} \langle x - y, Bx - By\rangle_- &= \langle x - y, \varepsilon Ax - \varepsilon Ay - (x - y)\rangle_- \\ &= \varepsilon\langle x - y, Ax - Ay\rangle_- - |x - y|^2 \\ &\leq (\varepsilon\omega - 1)|x - y|^2. \end{aligned}$$

Let x be in D and for each $h > 0$ let x_h be a member of D such that $d(x + hAx; D) \geq |x + hAx - x_h| - h^2$. Note that $\lim_{h\to 0+}|x + hAx - x_h|/h = 0$. Now define

$$y_h = (1 + \varepsilon)^{-1}[\varepsilon x_h + hw + (1 - h)x].$$

Since D is convex and x_h, w and x are in D, y_h is in D whenever $0 < h < 1$. Thus, if $k = (1 + \varepsilon)^{-1}h$ and $h \in (0, 1)$,

$$\begin{aligned} d(x + kBx; D) &\leq |x + kBx - y_h| \\ &= |[(1 + \varepsilon)x + h\varepsilon Ax + hw - hx] \\ &\qquad - [\varepsilon x_h + hw + (1 - h)x]|(1 + \varepsilon)^{-1} \\ &= \varepsilon(1 + \varepsilon)^{-1}|x + hAx - x_h|. \end{aligned}$$

Hence $\lim_{k \to 0+} d(x + kBx; D)/k = 0$; so B is the generator of a semigroup V of type $(\varepsilon\omega - 1)$ on D. Since $|V(t)x - V(t)y| \leq |x - y|e^{(\varepsilon\omega - 1)t}$ for each $t > 0$, $V(t)$ is a strict contraction from D into D, and hence there is a unique point x_t in D such that $V(t)x_t = x_t$. Since $V(s)x_t = V(s)V(t)x_t = V(t)V(s)x_t$, there is a unique point x^* in D such that $V(t)x^* = x^*$ for all $t \geq 0$. Consequently, $Bx^* = 0$ and $x^* - \varepsilon Ax^* = w$. Thus the range of $I - \varepsilon A$ contains D and we have that (ii) implies (iii). This completes the proof of Theorem 6.

Remark 4. If the set D is not convex, then it is not necessarily the case that (i) implies (iii) in Theorem 6. As a simple example, let E be the real Euclidean space R^2, let $D = \{(x,y) \in R^2 : x^2 + y^2 = 1\}$, and let $A(x,y) = (y, -x)$ for each $(x,y) \in D$. Then A is the generator of a semigroup U of type 0 on D (in particular, $U(t)(x,y) = (x \cos t + y \sin t, -x \sin t + y \cos t)$), but the image of D under $I - \varepsilon A$ does not intersect D for any $\varepsilon > 0$.

5. **Comments and examples.** The condition (C2) is intimately related to the notion of flow-invariant sets for differential equations. If E is finite dimensional, D is closed, A is defined on $[a, b) \times \Omega$ where $\Omega \supset D$ and Ω is open, and A satisfies a local Lipschitz condition on Ω, then Brézis [2] shows that D is invariant for (IVP) if and only if (C2) holds. Using the concept of a vector being normal to D at a point y of D, Bony [1] established a more subtle condition for D to be invariant for (IVP). For a discussion of these results, as well as some simplifications and improvements, see the paper of Redheffer [15].

One improvement employed by Redheffer is that the "lim" in (C2) is replaced by "lim inf". It is easy to check that Propositions 1 and 2 and Theorems 1 and 2 of this paper are also valid with "lim" replaced by "lim inf" (and with no change in proof). However, in the proof of Theorem 3, we used (C2) as it is (see, in particular, (2)′ in the proof of Lemma 1). In addition, Theorems 1, 2 and 3 of this paper give extensions of Theorems 4 and 5 of [15] to Banach spaces, and hence provide an answer to a question raised by Redheffer in [15]. Redheffer does use a more general type of dissipative condition. For example, the inequality in (3.5) of Theorem 3 should be $\langle x - y, A(t,x) - A(t,y)\rangle_+ \leq |x - y|\rho(|x - y|)$ where $\rho : [0, \infty) \to (-\infty, \infty)$ belongs to a certain class of "uniqueness functions". The proofs in this case may be carried through with only minor modifications—see [14].

Recently, M. Crandall [6] and P. Hartman [8] have shown that if E is finite dimensional and (Cl) and (C2) are fulfilled, then (IVP) has a solution. The techniques used in [6] and [8] are different from each other, and are also less tedious than those employed here. However, let us point out that by using (Cl) and (C2) and noting that the approximate solutions constructed in Proposition 1 are equicontinuous and uniformly bounded, the existence of a solution to (IVP) in the finite dimensional case follows easily from Proposition 2. This also extends to the infinite dimensional case if it is assumed that A maps bounded sets into relatively compact sets.

We now give some examples which connect our techniques with those of others, and also point out some applications of our results to fixed point theorems.

Example 1. Suppose that D is closed and convex, ∂D denotes the boundary of D, and A is a continuous function from $[a,b) \times D$ into E having the property that $A(t,x) + x \in D$ whenever $(t,x) \in [a,b) \times \partial D$. If $0 < h < 1$ and $x \in \partial D$, then $x + hA(t,x) = (1-h)x + h[A(t,x) + x] \in D$; so $\lim_{h\to 0+} d(x + hA(t,x); D)/h = 0$. If x is in the interior of D then obviously $\lim_{h\to 0+} d(x + hA(t,x); D)/h = 0$. Thus (C2) is satisfied and the results presented here contain some of those of M. Crandall [5].

Example 2. Suppose that D is closed, (Cl) holds, and that equation (3.11) in the statement of Theorem 4 is satisfied. For each $t \in [a,b)$ and $h > 0$ let $D_h(t)$ be the image of D under the mapping $I - hA(t,\cdot) : x \to x - hA(t,x)$, and define $J_h(t)x = (I - hA(t,\cdot))^{-1}x$ for each x in $D_h(t)$. Suppose also that for each $(t,x) \in [a,b) \times \partial D$, there is an $h_0 > 0$ such that $x \in D_h(t)$ for all $0 < h < h_0$. Using the techniques of T. Kato [11, §4], it follows that for each $(t,x) \in [a,b) \times \partial D$, $\lim_{h\to 0+} |J_h(t)x - x| = 0$, and hence $\lim_{h\to 0+} |A(t,J_h(t)x) - A(t,x)| = 0$. Noting that $A(t,J_h(t)x) = [J_h(t)x - x]/h$, we have that

$$\lim_{h\to 0+} d(x + hA(t,x); D) \leq \lim_{h\to 0+} |x + hA(t,x) - J_h(t)x|/h$$
$$= \lim_{h\to 0+} |A(t,x) - A(t,J_h(t)x)| = 0.$$

Thus we see that (C2) is also satisfied.

Finally, we point out how the results of §4 can be applied to the fixed point theory of nonlinear operators. These results are directly analogous to those of F. Browder [3] and M. Crandall [5].

Proposition 3. *Suppose that D is a closed subset of E and B is a continuous function from D into E such that*

(i) *there is a number $k < 1$ such that $\langle x - y, Bx - By\rangle_+ \leq k|x - y|^2$ for all $x, y \in D$; and*

(ii) $\lim_{h\to 0+} d(x + h[Bx - x]; D)/h = 0$ *for each $x \in D$.*

Then B has a unique fixed point in D.

Proposition 4. *Suppose that E is uniformly convex, D is a closed, convex subset of E, and B is a continuous function from D into E such that*

(i) *there is a number $k \leq 1$ such that $\langle x - y, Bx - By\rangle_- \leq k|x - y|^2$ for all $x, y \in D$;*

(ii) $\lim_{h\to 0+} d(x + h[Bx - x]; D)/h = 0$ *for each $x \in D$; and*

(iii) *$B - I$ is unbounded on unbounded subsets of D.*

Then B has a fixed point in D.

Note that, by the results of §4, if $A = B - I$ where B is as in Proposition 3 or 4, then A is the generator of a semigroup U of type $k - 1$ on D. Also, the fixed points of B are precisely the set of points x^* in D such that $U(t)x^* = x^*$ for all

$t \geq 0$. The proof of the existence of such points x^* in D is the same as in [3] and [5] and is omitted.

Added in proof. Professor James A. Yorke recently informed the author that conditions (C1) and (C2) were used by M. Nagumo (*Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys.-Math. Soc. Japan (3) **24** (1942), 551–559) to establish existence criteria in the case that E is finite dimensional. For further results using these conditions, see J. A. Yorke, *Differential inequalities and non-Lipschitz scalar functions*, Math. Systems Theory **4** (1970), 140–153, and references cited there.

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